# Magnetic Field of a Wire.
# The Method of Equivalent Sources

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Abstract.

**Objective.** To study the problem of magnetic field of a wire of arbitrary shape, carrying a direct current. It is shown that, in spite of formally correct construction, the result obtained by the method of Green functions in the particular case of circular current loop, is incorrect. Therefore, the first task was to work out an approach to this particular problem, and the second task was to generalize it to the general case of a wire specified by an arbitrary curve.
**Methods.** It turns out, that the problem of magnetic field of a circular current loop can be solved by the method of variables separation in toroidal coordinates, that reduces the equation for the magnetic field to an ordinary differential equation, which was solved analytically. The next task, to generalize this approach to an arbitrary curve, that leads to the problem of dividing a wire into circular segments and the corresponding division of the field. Solution of this problem is based on discovery of unusual sources whose magnetic fields are identically zero in certain domains of the space. This discovery allows to obtain the field of a wire composed of circular segments.
**Results.** It is shown by straightforward substitution that the expression for magnetic field, obtained by the method of Green functions, does not satisfy the Maxwell equations. Applicability of the method to Maxwell equations is analyzed and it is shown that justification of the method contains serious mistakes. The problem of the magnetic field of a circular current loop is solved in toroidal coordinates by the method of variables separation. A new kind of sources of stationary magnetic fields is discovered, which possess a strange property to produce the field only in a certain domain of the space, whereas in all the rest space the field produced by such a source is identically zero. The field of a wire decomposed into circular arcs is obtained by using the results obtained in toroidal coordinates. PACS 03.50.De

## Contents

# Introduction

## 1. The role of mathematical methods in physics

A typical physical theory starts with observations related to a certain class of phenomena, then formulates their main laws, first in the integral, second in the differentials form, and finally, derives the main equation. Examples of main equations of physical theories are well-known. So are Hamilton-Jacobi equation in mechanics of a mass point, Poisson equation in electrostatics, Schrodinger equation in quantum mechanics and so on. Solutions of this equation are believed to be in one-to-one correspondence with all phenomena which the theory describes, therefore, content of the latter depends to a large on exact solutions of their main equations. Theoretical physics of $XIX$ century contains a number of theories composed this way.

A physical theory is interesting for its ability to predict details of physical phenomena. Predictions follow from solutions of its basic equations. Unlike particular or numerical solutions, complete analytical ones provide entire description of phenomena with accuracy of the theory itself and allow all parameters to run the whole of their ranges. Completeness of descriptions of this sort reveals in freedom of choice of the values of all parameters in their continuous ranges, so that each solution actually provides description of a multi-dimensional continuum of phenomena of a given class. Evidently, all descriptions of this sort constitute an important part of the theory. Therefore, mathematical methods used for obtaining analytical solutions, constitute a necessary part of any theory. Contents of a physical theory strongly depends on availability of mathematical methods which can be used for these ends.

As the main equation is derived or postulated, a theory enters the new stage of its development, on which its main equation remains fixed, and the task is to obtain as much as possible solutions of the equation and use them to describe as much as possible phenomena. Standard texts on mathematical physics show what should be done as the main equation of a theory is known. The equations must be solved by using standard mathematical methods provided in the texts and solutions must be obtained in the most general form. However, sometimes the standard methods of mathematical physics do not work. This fact actually divides theoretical physics into two parts, one of which was successively developed with use of the standard methods of mathematical physics, and another was not. The earlier contains all linear theories and linearized versions of all the rest ones and the latter consists of only non-linear theories. Even some physical theories are divided into their original and linearized parts. For example, the whole variety of phenomena of gas dynamics, which is a non-linear theory, are divided this way into its linear part, restricted to the theory of weak waves propagation, and all the rest part of the subject. The earlier was successively developed with use of the commonplace methods of mathematical physics whereas the latter remains mostly the area of numerical simulations.

If the equation is linear, it can be solved in the most general form, that is an orthogonal expansion over the space of particular solutions. Theoretical physics of $XIX$ century was based at a large on mathematical methods for solving linear equations of mathematical physics. If a theory is linear the methods work properly and allow to describe and explain details of a broad class of physical phenomena. These descriptions provide all interesting details of phenomena with accuracy of theory itself and allow to make precise predictions which help to plan actions, design facilities, in other words use the theory. That is why physical theories are needed.

The difference between linear and non-linear theories is well-known. Linear theories describe phenomena which constitute linear spaces, therefore, mathematically, these phenomena are mapped into linear functional spaces of complete solutions of their main equations. As a result, the whole of linear space of the phenomena is described by the corresponding linear space of solutions of the main equation, that provides completeness of the description. Unlike them, non-linear theories reduce to non-linear main equations, which cannot be solved in the most general form. Therefore, non-linear theories describe only particular cases via particular solutions.

## 2. Magnetostatics and mathematical physics

Electromagnetic interaction is the most actively used kind of fundamental interactions, therefore, it always was extremely important to know exactly the form of electromagnetic fields produced under all possible conditions. It is absolutely clear, how to get this knowledge – just to solve the main equations of the theory as it was done in a number of physical theories. However, from the very beginning, development of this classical electrodynamics goes completely different way and presently its structure differs fundamentally from that of other linear theories. The only exception consists of electrostatics and a small part of magnetostatics, which reduce to the theory of simple and double layers correspondingly [**1**]. The main equation of these theories, which is the Poisson equation, was solved in the most general form in the same manner as all the rest equations of mathematical physics. All the rest part of the theory of electromagnetic fields has an absolutely different structure based on particular solutions obtained by the method of Green's functions. It was natural to try to shape the subject out as a standard linear theory, but for some reasons it was not done. In this book we complete this work in a small part of electrodynamics, which studies only stationary magnetic fields produced by filamentary currents, that is a special branch of magnetostatics. In other words, the main subject of this book is theory of magnetic field of a wire carrying a direct current.

Surprisingly enough, though magnetostatics is a linear theory, it is built at a large as if it was a non-linear one. Magnetic fields even for the simplest sources are found in terms of particular solutions, though the equations could well be solved in the most general form. This fact signifies that linear physical theories is also divided into two parts. One part consists of linear theories whose main equations are scalar and which therefore were successively developed by using the commonplace methods of mathematical physics. Another part contains non-scalar linear theories, which could not be developed this way. The point is that the methods worked out for solving scalar equations do not help to solve non-scalar ones, particularly, the Maxwell's equations.

Classical electrodynamics is apparently a non-scalar theory. Though Maxwell equations are linear, they are non-scalar and therefore remain actually unsolved till now. Even their simplified version found in magnetostatics, has not been solved systematically in all known coordinate systems, whereas the same work was completed for all scalar linear equations. Instead of searching for non-scalar extensions of the methods of mathematical physics, majority of physicists tried either to apply the scalar methods to the Maxwell's equations or construct particular solutions. Finally, the method of Green's functions became the main method in classical electrodynamics and even in magnetostatics of a thin wire. This activity actually stopped development of theoretical physics in this direction long ago. In this book we criticize the generally accepted approach to Maxwell equations and show by a direct verification that, most likely, all non-trivial results in electrodynamics, obtained by the method of Green's functions, are erroneous.

Since classical electrodynamics is a linear theory, solution of each its problem should be presented in the most general form, which is an orthogonal decomposition over the space of particular solutions of the Maxwell equations, as it is done in all linear theories, particularly, theories of small oscillations. Systematical solution of the stationary version of the Maxwell's equations would shape magnetostatics as a standard linear theory. It will be shown that this necessary work cannot be completed by the method of Green's functions. Only solutions of the equation as they are, obtained by the method of variables separation, can be useful.

### 3. Magnetic field of a wire carrying a direct current

In his monograph "Classical Electrodynamics" [**1**], J.D. Jackson wrote: "*The basic differential laws of magnetostatics are given by*

$$\begin{aligned} \nabla \times \vec{B} &= \frac{4\pi}{c}\vec{J} \\ \nabla \cdot \vec{J} &= 0. \end{aligned} \tag{3.1}$$

*The problem is how to solve them. If the current density is zero in the region of interest,* $\nabla \times \vec{B} = 0$ *permits the expression of the vector magnetic induction* $\vec{B}$ *as the gradient of a magnetic scalar potential,*

$$\vec{B} = -\nabla\Phi_M.$$

*Then* (3.1) *reduces to Laplace's equation for* $\Phi_M$, ..." In the present book we consider, probably, the most important part of magnetostatics with the current density being equal to zero everywhere but a curve which represents a filamentary wire.

The equations (3.1) look similar to commonplace equations of mathematical physics with the only difference that they are non-scalar. Taking divergence of both sides of the first of them shows that divergence of the vector $\vec{J}$ is identically zero. This property of the current density exposes the electric charge conservation law. Due to this law, a filamentary wire cannot have endpoints, thus, either is closed or disappears at infinity. Otherwise, an electric current requires that electric charge appears in one endpoint and disappears in another. Then the current density has non-zero divergence, particularly, in the endpoints of the curve that would make the equations (3.1) inconsistent. So, they have no solutions with a right-hand side of this form. This fact exposes the main difference between electrostatics and magnetostatics. Electrostatics possesses the notion of the simplest and the most fundamental source of the field, which is a point-like charge. All the rest sources in electrostatics can be represented with arbitrarily high accuracy as collections of point-like charges. Unlike it, magnetostatics of filamentary wires does not possess a certain "the most fundamental" source of the field which would play the role of an elementary part of an arbitrary curve because an arbitrary curve cannot be represented as a collection of "more elementary" endless curves. Nor it can be represented as a collection of segments due to the second equation (3.1) because a segment of a curve has endpoints. Therefore, an endless filamentary wire itself plays the role of the most fundamental source of magnetic field. This kind of source is rather a concept than a physical object and this concept is the whole variety of all possible endless curves. Since each curve produces its own field, it is important to find a more or less general approach to the equations (3.1) with filamentary current in the right-hand side.

There exist a lot of problems of this kind in mathematical physics. Usually, the task is to find out a scalar potential produced by a source of a simple geometric form like a sphere, disk, cylinder or similar. The task reduces to a boundary problem with boundary conditions specified on a surface of second order [**2, 3**]. Mathematical methods for solving these equations are well-known, however, in fact, all complete analytical solutions of these problems are obtained only in some certain coordinate systems, composed mostly of surfaces of second order, in which the boundary coincides with one of these surfaces. Thus, complete solutions are obtained mainly in cases when the boundary conditions are specified on a surface of second order. Problems with boundaries of this kind are solvable because there always exists such a system that the boundary can be specified by fixing only one of coordinates. However, the variety of surfaces of second order meets broad demands of practice.

Appearance of a curve as a boundary is somewhat special and requires the curve to coincide with a limiting case of a coordinate surface. This happens in some coordinate systems, for example, the axis $\rho = 0$ in round cylinder coordinates $\{z, \rho, \varphi\}$ is a limiting case of coordinate cylinders $\rho = \text{const}$, but all curves appeared this way in usable coordinate systems, also are of second order. Therefore there are no usable coordinate systems in which an arbitrary curve can appear as a limiting case of a coordinate surface, so, the problem under consideration differs from those found in standard texts on mathematical physics. Solutions of this problem found in standard texts on classical electrodynamics are related only to the simplest case when this curve is either a straight line or a circle. In this book, we go further and work out an approach which makes it possible to obtain magnetic field of a wire specified by a curve composed of circular arcs. Our method is based on our discovery of equivalent sources of magnetic fields. This discovery allows us to replace a circular arc with another object which produces exactly the same magnetic field, so, if a wire is composed of circular arcs, it can be replaces by a chain of its equivalents that, after all yields the field of the wire.

In this book, the following notations and agreements are used. The fields are considered in vacuum, therefore, we make no difference between strength $\vec{H}$ and induction $\vec{B}$ and use only the earlier. Another feature of this book is that sometimes we employ the calculus of exterior differential forms, particularly, when reducing the equations (3.1). In fact, magnetostatics provides an excellent illustration of this kind of mathematics. Besides, the calculus of exterior differential forms allows us to explain an important difference between two distinct kinds of sources which have the form of a curve. This difference plays an important role in our considerations and exposes an interesting mathematical fact. It turns out that there exist two distinct kinds of 2-dimensional $\delta$-functions in a 3-space which correspond to two distinct kinds of filamentary sources of fields. To show this, we propose some modification of the notion of $\delta$-function, which is presented in the next section.

## 4. The simplest cases

In the simplest cases a source of the field possesses the highest symmetry and so does the field produced by it. The most symmetric curves are a straight line and a circle. The earlier possesses cylindric symmetry generated by a 2-parameter group of transformations and the latter does only the 1-parameter group of rotational symmetry. There exists one more kind of curves possessing a symmetry, whose transformations are fixed combinations of shifts and rotations (so-called screw symmetry), a spiral, which corresponds to an important source of magnetic fields called solenoid. However, the task to construct magnetic field of solenoid will not be considered in this book for the following reason. There is a significant difference between the screw symmetry and the symmetries of translations and rotations. These two symmetries admit usage of parameters of their transformations as coordinates in the space. Usually we denote them as $z$ and $\varphi$ so that these symmetry transformations are $z \to z + \text{const}$ (translation) and $\varphi \to \varphi + \text{const}$ (rotations) correspondingly. Unlike them, the screw symmetry does not admit usage of its transformation parameter as a coordinate, therefore, even if a source and its field possess this symmetry, this fact does not help to solve the field equation. As a result, in spite of symmetry this problem is as difficult as the most general one.

In the case of a straight line it is convenient to introduce a scalar magnetic potential $\Phi_M$ in round cylinder coordinates $\{z, \rho, \varphi\}$, whose gradient is directed everywhere around the $z$-axis. Rotational symmetry of the field signifies that the gradient does not depend on the coordinate $\varphi$. The only possible form of the potential possessing these properties and satisfying the Laplace equation, is $\Phi_M = C\varphi$. Consider integral of its gradient $\mathrm{d}\varphi$ over an arbitrary simple contour in the space. By construction, if the contour does not surround the wire, the integral is equal to zero. If the contour surrounds the axis $\rho = 0$, the integral equals $2\pi C$. This fact signifies that if $\mathcal{I}$ is magnitude of the current in a straight line wire, its magnetic field is equal to $2\mathcal{I}\nabla\varphi$.

If instead of a scalar magnetic potential we use the vector potential, it is convenient to represent it as a 1-form $\alpha = A_i\,\mathrm{d}x^i$ which in round cylinder coordinates has single non-zero component $A_z$. Due to the symmetries of the problem, this component depends only on the variable $\rho$, then the strength of the magnetic field is a 2-form

$$\mathrm{d}\alpha = -\frac{\mathrm{d}A}{\mathrm{d}\rho}\,\mathrm{d}z \wedge \mathrm{d}\rho$$

and the equation (1.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}A}{\mathrm{d}\rho}\right) = \frac{4\pi}{c}\,I$$

where the right-hand side is zero everywhere but the axis $\rho = 0$. The only solution of this equation has the form $A(\rho) = -2\mathcal{I}\ln\rho\,\mathrm{d}z$ that corresponds to the same strength, because

$$\mathrm{d}\alpha = \frac{\mathcal{I}}{\rho}\,\mathrm{d}z \wedge \mathrm{d}\rho, \quad {}^*\,\mathrm{d}\alpha = 2\mathcal{I}\,\mathrm{d}\varphi.$$

The case of a circular current loop is more complicated. There exists a coordinate system in which the loop coincides with a limiting case of a coordinate surface, hence, can be specified by a single equation. This system will be described in details in the next chapter. In these coordinates the problem of magnetic field of a circular current loop can be solved both for the scalar magnetic potential and for the vector potential. This will be done in the Chapters 1 and 2, where first, the coordinate system will be introduced, then the equations (1.3) will be reduced to ordinary differential equations and solved analytically. Below, we analyze the generally accepted expression for the field obtained by the method of Green function.

## 5. Magnetic field of a circular current loop: the method of Green function

The vector potential of a circular current loop obtained by the method of Green function in spherical coordinates is presented in the book [**1**], equation

(5.36), by the expression

$$A_\varphi = \frac{I}{ca}\int\limits_0^{2\pi} \frac{\cos\phi'\,\mathrm{d}\phi'}{(a^2+r^2-2ar\sin\theta\cos\phi')^{1/2}}. \tag{5.1}$$

It must be pointed out that there exist certain conditions, which the $\varphi$-component of any continuous vector field must meet. First, as seen from geometric considerations, if a vector field is continuous, its $\varphi$-component is zero on the axis $\sin\theta = 0$. Second, if the vector field has continuous curl, its derivative in the orthogonal direction to the axis, also is identically zero. This condition will be considered more thoroughly in the next chapter. In fact, the vector potential (5.1) satisfies none of these conditions therefore it is neither continuous, nor has continuous curl on the axis. Should a circular current loop produce the vector potential of this form, as J.D. Jackson claims, its double curl would be zero almost everywhere. Let us look, however, whether it is so. Explicit form of the curl of an arbitrary vector field in these coordinates is found in the J.A. Stratton's book [**5**], equation (95):

$$\begin{aligned}\nabla\times\vec{F} = {} & \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\cdot F_3) - \frac{\partial F_2}{\partial\varphi}\right]\vec{\imath}_1 + \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_1}{\partial\varphi} - \frac{\partial}{\partial r}(rF_2)\right]\vec{\imath}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_2) - \frac{\partial F_1}{\partial\theta}\right]\vec{\imath}_3\end{aligned} \tag{5.2}$$

where $\imath_k$ are unit vectors tangent to coordinate lines in each point. Matching the notations used in these two books shows that Jackson's $A_\varphi$ corresponds to the component $F_3$ of the Stratton's book and direct substitution shows without the entire calculation, that the vector potential (5.1) has non-zero double curl everywhere. It is convenient to denote $A = F_3 r\sin\theta$, then,

$$\nabla\times\vec{F} = \frac{q}{r^2\sin\theta}\frac{\partial A}{\partial\theta}\vec{\imath}_1 - \frac{1}{\sin\theta}\frac{\partial A}{\partial r}\vec{\imath}_2.$$

Substituting the result into the equation (5.2) yields explicit form of the double curl of the vector potential:

$$\nabla\times\nabla\times\vec{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial A}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial A}{\partial\theta}\right)\right]\vec{\imath}_3,$$

thus, the equation for the function $A$ has the form

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A}{\partial\theta}\right) = 0. \tag{5.3}$$

Evidently, the second term contains a $\cot\theta$ singularity which the first term has not, consequently, their sum cannot be zero. As a result, the generally accepted expression for the vector potential (5.1) provided in the J.D. Jackson's book does not satisfy the equation (3.1) and actually, corresponds to some non-zero density of current in the space. Further, in the equation (5.38), J.D. Jackson provides all components of the vector potential and induction, among which there are

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi), \tag{5.4}$$

and an approximate expression

$$A_\varphi(r,\theta) = \frac{I\pi a^2}{c}\,\frac{r\sin\theta}{(a^2+r^2+2ar\sin\theta)^{1/2}} \tag{5.5}$$

in the equation (5.39) and, finally, the corresponding expression for the induction components. One of them is represented as

$$B_r = 2\left(\frac{I\pi a^2}{c}\right)\frac{\cos\theta}{r^3}.$$

This result is very strange. The expression could be obtained from the approximation (5.5) by differentiating it, but, besides, the result should also be divided by $\sin\theta$ as shown in the equation (5.4). However, even without any calculations it is quite clear that if everything is done as in the equation (5.4), the result possesses a singularity under $\sin\theta = 0$. In other words, if everything is done as in the equation (5.4) the result would be

$$B_r = 2\left(\frac{I\pi a^2}{c}\right)\frac{\cot\theta}{r^3},$$

hence, if the vector potential is obtained by the method of Green functions, then $B_r$ contains the factor $\cot\theta$. This singularity remains in the result, because another component has no such a factor. In any case, the double curl of the induction obtained by the method of Green functions, is not zero beyond the current loop, as expected. In other words, the method of Green's functions yields wrong results, one of which is the expression for the vector potential presented in the J.D. Jackson's book. To obtain the correct expression for the field of circular current loop we need to solve the equation for the function $A$ which has the form (5.3) This will be done in the Chapter 3 in toroidal coordinates.

In the most general case when the wire has the form of an arbitrary curve, there are no usable coordinate systems in which the curve can be represented by a single equation. The problem how to solve the equations (1.3) becomes more difficult and it is not clear, whether solutions obtained in the two simplest cases can help to solve it. In other fields of theoretical physics, if a problem is solved for a straight line and circumference, the general solution can also be obtained by approximating a general curve by a collection of small round arcs. In magnetostatics such a division seems to be impossible because no solutions of the equations (3.1) exist, in which the current density is non-zero only in a segment. On the other hand, there is no other opportunity to represent the field of an arbitrary wire but to approximate this wire as a chain of circular segments. Each segment contributes the entire field of the wire, but this contribution cannot be obtained as an independent solution of the Maxwell equations. Nevertheless, contribution of a given segment of a wire will be determined and used to decompose the field of a wire into contributions of its segments.

## 6. Brief contents of this book

The present book is organized as follows. In the Chapter 1 we discuss foundations and applicability of the method of Green's functions to non-scalar covariant equations. In the course of these considerations we show that this method cannot be applied to the equations for magnetic field (3.1). In the Chapter 2 we present the main mathematical tools used below. One of them is construction of the toroidal coordinate system started from the scratch. This system will be used in the next chapter when solving the main equation. Another is the notion of equivalent sources of magnetic fields. Existence of classes of equivalent sources is a discovery which is presented in this book for the first time. The phenomenon of equivalence makes it possible to replace a segment of wire with a more convenient equivalent to it and determine this way contribution of a segment to the entire field. This method will also be used in the next chapter for constructing the field of a wire composed of circular arcs. One more mathematical tool presented in this chapter is representation of magnetic field produced by an arbitrary surface current density on a plane. This representation is a necessary part of the method of equivalent sources. In the Chapter 3 we reduce the equations (1.3) in the relevant toroidal coordinates to an ordinary differential equation and obtain an analytical solution which describes magnetic field of a circular current loop. This solution allows to find an equivalent of a circular segment which is a wedge composed of two conducting half-planes orthogonal to the segment in its endpoints. Equivalence of a circular segment and the corresponding wedge makes it possible to obtain one more class of strange sources of magnetic fields, which produce the field only in the wedge. Since such a source is an equivalent of a segment of a circular current loop, a wire composed of circular segments is equivalent to collection of half-planes carrying known current densities and finally, the field of such a wire can be constructed. In the Chapter 4 we show how to match two strange sources and their fields that provides an analytical representation of the field of a wire consisting of finite length segments. However, since the expression obtained this way is too complicated we finish all calculations and only outline the operations which can be completed numerically that provides the final result.

In this book we do not try to derive analytical expressions which describe completely the field of a given wire. We tried to divide the problems into parts and do what is possible in the framework of purely analytical and geometrical considerations. Out final goal was to create an analytical support for a future numerical solution of the problem. Our geometric considerations led to discovery of equivalent sources of magnetic fields which allow to replace a segment of a wire with a pair of conducting half-planes carrying certain surface current densities. Wedges built of half-planes are convenient parts which can be applies to each other and specify a certain division of the space. This division of the space by wedges, in which the field is known, constitutes the main idea of our approach to the problem. So, in fact, our goal is to expose our method of equivalent sources. Thus, the main goal of the present book is to expose the method of equivalent sources.

CHAPTER 1

# The method of Green functions

## 1. The method of Green function in magnetostatics

The method of Green functions provides solutions of numerous problems of mathematical physics, particularly, of those reduced to equations for scalar fields [**3**]. If the field is vectorial, the method needs an additional justification. Since a vector is referred to the local frame in each point, the corresponding Green function is referred to local frames in two distinct points. Usually, the frame in each point is specified by directions of coordinate lines or coordinate surfaces passing through it, therefore, in general, they differ from one point to another. By definition, a the Green function for the vector potential specifies its components in a given point, as if they are produced by a given component of the current in another point. Evidently, this definition employs the notion of the vector of "element of current", which, in fact, cannot be a source of magnetic field because it has endpoints. This fact alone signifies that the method of Green functions cannot be applied to this problem. Besides, such a two-point function must contain information about relations of local frames established in these points. Such a relation exists only in the flat space which admits distant parallelism that allows to build globally parallel fields of local frames. In a globally parallel field of local frames the Green function for a vector field is equal to the commonplace scalar Green function multiplied by the Kronecker $\delta$. The proof found in the M. Born and E. Wolf monograph [**4**], consists in the following. In standard Cartesian coordinates, the operation $\nabla \times \nabla \times$ (double curl) can be replaced by the Laplacian due to the identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \triangle \vec{A}, \tag{1.1}$$

proviso that the vector has zero divergence. The proof of this identity is presented only in Cartesian coordinates which correspond to a globally parallel field of local frames. In any other coordinate system the natural field of local frames is not globally parallel and this proof is broken. However, the same can be done in cylindric coordinates only if the operations are applied to the component $A_z$ of the vector, because this component is, actually, a Cartesian one, whereas two other components are not. For example, since the $\varphi$-component apparently is not a Cartesian one, the equality (1.1) is not valid for it.

In the case of a circular current loop the vector potential is strictly azimuthal, therefore, the scalar Green function, which is equal to inverse distance, cannot be applied. Nevertheless, the expression for magnetic field produced by this source, obtained by the method of Green function and found in numerous texts and monographs, for example, in J.D. Jackson' "Classical Electrodynamics" [**1**], is generally accepted. Therefore, the result obtained this way needs to be verified. The simplest way to do it, is to substitute the expression for the magnetic field into the equation (0.3.1).

## 2. Scalar and non-scalar equations

Solving a typical problem of mathematical physics in a standard Cartesian coordinates is a quite simple procedure which does not require any special methods. Methods of mathematical physics have been worked out with a purpose to solve physical problems in all the rest coordinate systems. Variety of usable coordinate systems is needed to simplify solution of problems which cannot be solved in Cartesian coordinates. For example, this system is not used when describing a field of a point-like or axially-symmetric sources. Evidently, the whole of variety of coordinate systems used in the scalar part of mathematical physics, would be useful in classical electrodynamics as well.

However, standard texts on mathematical physics, which teach how to solve scalar equations, do not explain how to solve non-scalar equations encountered in magnetostatics. As a rule, when trying to solve a non-scalar equation, the first step an ordinary physicist makes, is an attempt to replace it with a scalar one. The equations (0.3.1) reduce to a single equation by introducing the vector potential $\vec{A}$ so that the second equation is completed automatically and it remains to solve the equation

$$\nabla \times \nabla \times \vec{A} = \frac{4\pi}{c}\vec{J}. \tag{2.1}$$

The next step is to determine properties of the double curl operator in the left-hand side of this equation. These properties have been studied properly only in standard Cartesian coordinates [**4**]. As was pointed out above, in standard Cartesian coordinates, the operation $\nabla\times\nabla\times$ (double curl) can be replaced by the Laplacian due to the identity (1.1). This identity is proved with use of the fact that the Levi-Civita symbol $\varepsilon_{ijk}$ which presents in the curl operator, is constant in Cartesian coordinates. In any other coordinate system it is not so and the identity is broken. This fact signifies that double curl can be replaced by Laplacian only in a standard Cartesian coordinate system, besides, in all cylindric coordinates it can be applied only to the component $A_z$ of a vector, because this component is, actually, a Cartesian one. In case case of $\varphi$-component whose divergence also is zero, its minus double curl is not equal to its Laplacian because $\varphi$-component of a vector is not a Cartesian one.

Application of the scalar Laplacian a to a vector rises more questions. First of all, operators which appear in co-variant equations, are co-variant themselves. Their covariance means that they are defined disregard of choice of a coordinate system. Operators, defined in coordinate-free form are well-known, for example, so are gradient, curl or divergence and so is Laplacian: Laplacian of a scalar is divergence of its gradient. A vector has no gradient, consequently, in general, the scalar Laplacian, as it was defined, cannot be applied to a vector or its components. If it is applied to a component of a vector as if this component was a scalar, then the operation is non-covariant and its result depends on choice of a coordinate system that is apparently wrong. On the other hand, non-covariant operations cannot appear in field theories like magnetostatics. In other words, the identity (1.1) is non-covariant and hence, cannot be valid in general. In standard Cartesian coordinates this identity is in force, but this coordinate system itself exists only in Euclidean space, therefore usage of this identity encapsulates an assumption that the space is flat. It must be noted, however, that there exists only one physical theory called general relativity, which prescribes the space, what geometry it may have. All the rest theories are to be valid disregard of any assumptions about the space. So, if magnetostatics obeys this principle, no its part can be based on the identity (1.1).

## 3. Green functions for non-scalar equations

Since the equations (0.3.1) reduce to the Laplace equation, all the rest could well be done as in a standard linear theory. Solutions of the equations could be obtained in their general form and the method of Green functions, which yields only particular solutions, would not appear in magnetostatics at all. Nevertheless, the only attempt to find solution of the simplest problem of this theory, namely, that of the field of a circular current loop was made by use of the method of Green functions. It looks like that physicists did understand that replacing of double curl with the scalar Laplacian cannot be made for the $\varphi$-component of the vector potential, therefore they did not try to obtain this component from solutions of the Laplace equation by the method of variables separation. Instead, they actually did the same by the method of Green functions.

Since the identity (1.1) is valid only in Cartesian coordinates, the method of Green functions can be used only in standard Cartesian coordinates too. However, a physical theory cannot be related to a certain coordinate system or a class of coordinate systems. Consequently, the method of Green functions cannot play any important role in a non-scalar physical theory, particularly, in magnetostatics. Nevertheless, it is this method which was applied to numerous problems of classical electrodynamics without necessary justification.

The next question is, how to define the Green function for the equation (2.1) using the identity (1.1). The first consequence of this identity is that the vector of current produces vector potential strictly parallel to it everywhere in the surrounding space. Again, we see how the method used prescribes the space what geometry it must have. The point is that two vectors taken in two distinct points can be (or not be) parallel to each other only in Euclidean space. However, if this prescription is accepted, we have an opportunity to define the Green function for the equation (2.1) as the scalar Green function $G(\vec{r},\vec{r}') = |\vec{r}-\vec{r}'|^{-1}$ multiplied by the Kronecker $\delta$:

$$G_{ij}(\vec{r},\vec{r}') = \delta_{ij}G(\vec{r},\vec{r}').$$

This Green function can be used for obtaining vector potentials of a straight current, particularly, in case of an infinite straight wire in Cartesian and round cylinder coordinates. In case of a circular current loop the vector potential has only $\varphi$-component in a relevant (round cylinder, spherical or toroidal) coordinate system. Since the identity (1.1) is not valid for a non-Cartesian components of a vector, the vector potential of a circular current loop cannot be obtained by the method of Green function. Nevertheless, the expression for the vector potential obtained by this method became generally accepted mainly thanks to the well-known monograph of J.D. Jackson [**1**]. Below we analyze critically applicability of the method of Green's functions to the problems of magnetostatics.

## 4. Continuity of an azimuthal vector field

Some properties of the field of a filamentary circular current loop are evident. The field possesses axial symmetry, its vector potential is purely azimuthal, in other words, it has only one non-zero component $A_\varphi$, the vector potential and the strength are continuous everywhere but the loop itself. The most convenient coordinate systems to describe axially-symmetric objects are those with the azimuthal angle $\varphi$ as one of coordinates. If such a system is chosen properly, any axially-symmetric object can be presented in it by using only two dimensions. An azimuthal vector field $\vec{A}$ which does not depend on $\varphi$, is an axially-symmetric object, hence it can be represented in such a coordinate system in terms of functions of two variables. Let us analyze now what does it mean that such a field is continuous everywhere but the loop, particularly, on the axis of symmetry.

First of all, we need to clear up what is the meaning of the component $A_\varphi$, what is the difference between it and $A^\varphi$ and what is norm of a vector presented by its $\varphi$-component while two other components are zero. It was explained in numerous books how to handle components of a vector in a arbitrary (non-Cartesian) coordinate system. It was done also in the J.A. Stratton monograph on electrodynamics [**5**] specially to teach the readers to understand electromagnetic fields beyond the standard Cartesian coordinates. The answer to the questions posed above consists in the following. In a coordinate system with $\varphi$ being one of coordinates the metric components $g_{\varphi\varphi}$ and $g^{\varphi\varphi}$ are equal to $\rho^2$ and $\rho^{-2}$ respectively, where $\rho$ is the distance from the given point to the axis of symmetry. Then, the norm of the vector is $|\vec{A}| = \rho A^\varphi = \rho^{-1} A_\varphi$, consequently, none of the components $A_\varphi$ and $A^\varphi$ coincides with the norm of the vector.

Now, we can pass to the question, under what conditions a toroidal vector is continuous. A vector field is continuous if so is its $\varphi$-component. However, this condition is not sufficient on the axis itself. To be continuous on the axis, an azimuthal vector must have zero length there because in each point of this straight line this vector takes all possible directions orthogonal to it. Since we are discussing magnetic field of a circular current loop, the vector potential must meet also another condition, which reads that its curl is also a continuous vector field in the space. To see whether a strictly azimuthal vector potential meets this condition, we need to calculate its curl.

This calculation is based on the definition of the curl operation which consists in the following. Consider a small piece of plane and let a closed curve $C$ be its boundary. The curl of a vector is equal to the limit of a contour integral of this vector over the curve $C$, divided by the area of the piece of plane when the area tends to zero. Let the plane be orthogonal to the axis and the contour $C$ be a circle in form, with its center lying on the axis. Then it remains to integrate the norm of the vector over the single variable $\varphi$ from 0 to $2\pi$ that is trivial. Indeed, the norm of the vector equal to $\rho A^\varphi$ which does not depend on $\varphi$, therefore integration over this variable from 0 to $2\pi$ reduces to multiplication the integrand by the range equal to $2\pi$. Hence, the integral is equal to $2\pi\rho|\vec{A}|$ and it remains to divide it by the area of the circle. The area of the circle is $\pi\rho^2$, consequently, division by it yields the value $2\rho^{-1}A^\varphi = 2\rho^{-2}|\vec{A}|$. Passage to the limit $\rho \to 0$ yields an infinite result unless in the neighborhood of the axis of symmetry the norm of the vector $\vec{A}$ behaves as $\rho^2$. Hence, if the vector potential of magnetic field produced by a circular current loop has finite curl its norm behaves as

$$|\vec{A}| \sim \rho^2. \tag{4.1}$$

## 5. The method of Green functions for scalar equations

The method of Green functions is applied successively only to linear scalar equations of mathematical physics. The typical equation of mathematical physics has the form

$$\hat{L}\Psi = f, \tag{5.1}$$

where $\hat{L}$ is a linear operator. The main equations of mathematical physics are co-variant, so, their explicit form contains no information about geometry of the space and the coordinate system used. Therefore, so is the operator $\hat{L}$. All operators, which meet this condition, are known, in general such an operator is a linear combination of Laplacian, time derivatives and a constant. The method consists in the following. The Green function for the equation is a two-point function $G(P, P')$ which satisfies the equation[1]

$$\hat{L}\, G(P, P') = \delta(P, P'). \tag{5.2}$$

Co-variance of the equation entails special properties of this function. In a symmetric space it contains nothing but the distance between the two points. If this function is known, a particular solution of the equation (5.1) can be obtained in the form of the direct integral transformation of the source $f(P)$ with the Green function as the kernel:

$$\Psi = \int G(P, P')\, f(P')\, \mathrm{d}P'.$$

All expressions have a simpler form if it is postulated that the space is Euclidean and can be replaced with the vector 3-space $\mathbb{V}^3$ in which the notion of point is replaced with that of vector. In this space each two vectors $\vec{r}_1$ and $\vec{r}_2$ specify the third vector which is their difference $\vec{r}_2 - \vec{r}_1$, the spatial $\delta$-function is defined as the $\delta$-function of this difference that reduces to the product of 1-dimensional $\delta$-functions of differences of all Cartesian coordinates. Correspondingly, if the equation (5.1) does not contain coordinates explicitly, then, due to the symmetry considerations, the Green function is a function of single argument $\vec{r}_1 - \vec{r}_2$. This construction is used in standard Cartesian coordinates, but in any other coordinate system the $\delta$-function and the Green functions cannot be introduced such a simple way. Hence, the method of Green functions for a co-variant scalar equation has two different forms, one in standard

[1] here we return to the standard definition of the two-point $\delta$-function

Cartesian coordinates and one for all the rest coordinate systems. The method can be used actually only in Cartesian coordinates. Among other coordinate systems there are only two exceptions, namely, spherical and round cylinder coordinates, in which the method can be used only in special case when the source is punctual and is placed in the origin of coordinates. Usability of the method of Green functions for non-scalar equations is a more difficult problem. It will be considered in the next section.

## 6. The method of Green functions for non-scalar equations

The idea that a relevant particular solution of the equation (2.1) can obtained by a direct integral transformation of the source of the field, remains valid for non-scalar equations, but the problem is how to define the Green function. The first question is, what form the equation for the Green function of a non-scalar equation has. In other words, what form its right-hand side has, because it is not enough to have just the $\delta$-function defined via its integrals. In this case both the source and the field are not scalars, thus, a non-scalar source in one point produces a non-scalar field in another point of the space. Basically, in geometry we have no univalent relation between directions in two arbitrary points, unless the space is Euclidean. Hence, justification of the method of Green functions for a non-scalar equation is practically possible only under assumption that the space is Euclidean, whereas it remains unknown, whether the method can be justified otherwise.

To implement the idea of direct transformation of the source of the field into the field itself, one needs the kernel of this transformation, called the Green function. This kernel can be obtained only as a solution of the corresponding equation, which in the case of scalar field has the form (5.2). Hereafter we call it "the Green's equation". In general, it is easier to solve the original field equation than to find the Green function for it, and only in some particular cases the method of Green functions is useful. Nevertheless, let us return to the problem of definition of the Green function for a non-scalar equation.

Since, by definition, the Green function is a solution of the corresponding equation, this equation is the underlying base of the method. The only unknown detail of this equation is its right-hand side, though it is quite evident that this expression contains the $\delta$-function $\delta(P, P')$ as a factor. Another factor in the right-hand side of the Green's equation specifies a correspondence between spatial directions taken in two distinct points. The only possible correspondence between directions chosen in two distinct points is parallelism which exists only in Euclidean space. Since such a correspondence is to be specified in the foundations of the method for a non-scalar equation, flatness of the space also is a fundamental base of the method. Moreover, there is only one opportunity to use it. This opportunity is to suppose that the field produced by a point-like source is strictly parallel to it everywhere. Evidently, such an assumption is nonsense in any geometry but that of the Euclidean 3-space.

It is evident that Cartesian coordinates play their fundamental role in foundations of the method twice. First, they are needed to establish the first factor in the right-hand side of the Green's equation, which is $\delta(P, P')$, second, they specify the second factor, which is $\delta_{ij}$. Indeed, if the source produces the field strictly parallel to it everywhere, each Cartesian component of the source produces only the same component of the field in each point of the space. In other words, the second factor in the right-hand side of the Green's equation is found thanks to parallelism of local frames specified by this particular coordinate system.

Thus, thanks to specific properties of Cartesian coordinates, the form of the right-hand side of the Green's equation is cleared up. After that the equation can be solved that yields the desired Green function. As a result, the method can be applied to non-scalar equations under an additional assumption that the space is Euclidean and no other coordinate systems than Cartesian ones can be used. However, as was pointed out above, the method can be applied to scalar co-variant equations also in some exceptional cases in other coordinate systems. So, the question remains, whether there exist a similar exception for a non-scalar equation.

As was pointed out above, Green functions for co-variant scalar equations depend only on distance between the two points. This fact allows to employ the method when the source is point-like and placed in the origin of coordinates in a coordinate system in which the distance from the origin has a simple form. For instance, in round cylinder coordinates $\{z, \rho, \varphi\}$ the distance from the origin is $\sqrt{z^2 + \rho^2}$. At the same time, the origin is a singularity of the coordinate system, in which some directions cannot be specified. The only direction which can be specified in this point, is the $z$-direction, so, if there is a point-like non-scalar source, the coordinate system must have a certain orientation with the $z$-axis collinear with the source. However, such a source cannot be substituted into the equation (0.1.3) because it corresponds to a segment, hence, its divergence is non-zero. This fact signifies that the method of Green functions cannot be applied to these equations in coordinate systems other than Cartesian one. Therefore we conclude that the method of Green functions is completely useless when solving non-scalar field equations and all non-trivial results obtained with use of this method are wrong. Hereafter we consider only one approach to equations of this sort, based on the method of variables separation.